On the second largest eigenvalue of certain graphs in the perfect matching association scheme

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PIMS

Discrete Mathematics Research Group at the University of Regina

The spectrum of a graph

Definition

The **spectrum** of a graph G on n vertices is the spectrum of its adjacency matrix: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

Definition

The **spectral gap** of a graph G is defined as $\lambda_1 - \lambda_2$.

Motivation

The spectral gap of a k-regular graph is also known as its algebraic connectivity and corresponds to the smallest non-zero eigenvalue of the Laplacian matrix.

• Graphs with small spectral graphs tend to have large diameter.

• A large spectral gap implies stronger expansion properties and faster mixing of random walks on the graph.

Association schemes

Definition

Given a set of v points, a set $\mathcal{A} = \{A_0, A_1, \ldots, A_t\}$ of $v \times v$ binary matrices is an **association scheme** if:

- $A_0 = I_v$ (the identity matrix);
- $\sum_{i=0}^{t} A_i = J$ (J is the all-one matrix);
- $A^T \in \mathcal{A}$; (A^T is the transpose)
- $A_i A_j = c_o A_0 + c_1 A_1 + \ldots + c_t A_t$, where $c_i \in \mathbb{C}$;
- $A_i A_j = A_j A_i$ (matrices commute).

The indices of the scheme are known as the **relations or associates** of the scheme. An association scheme is **symmetric**, if $A_i = A_i^T$ for all relations.

Perfect matching

Definition

A **perfect matching** in a graph G is a matching that covers every vertex of G.

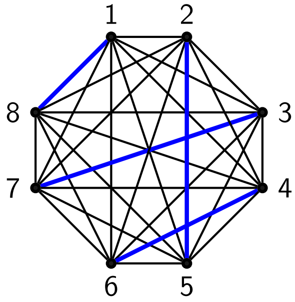

Figure: A perfect matching of K_8 (in blue).

Perfect matchings of K_{2n}

Definition

Let $M(K_{2n})$ denote the set of all perfect matchings of K_{2n}. An elementary counting argument will show that:

$$|M(K_{2n})| = (2n-1)(2n-3)\cdots(3)(1) = (2n-1)!!$$

Main goal: To construct the perfect matching association scheme in relation to $M(K_{2n})$.

Relation between two perfect matchings

We define a relation between two perfect matchings in $M(K_{2n})$.
Example: We overlap two perfect matchings of K_{2n}.

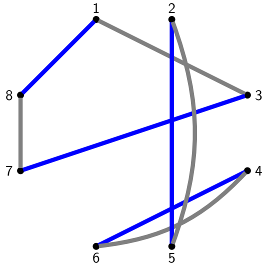

Figure: Two perfect matchings of $M(K_8)$ in grey and blue.

Relation between two perfect matchings

We define a relation between two perfect matchings in $M(K_{2n})$.
Example: This gives rise to a set of cycles of **even** lengths.

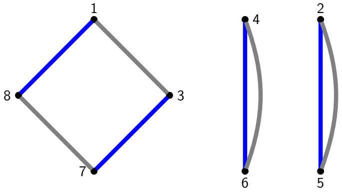

Figure: The union of these two matchings gives us 3 cycles of length 4,2, and 2 respectively.

Relation between two perfect matchings

Notation

Let $\mu \vdash n$ be a partition of n such that $\mu = [\mu_1, \mu_2, \ldots, \mu_t]$. We write $2\mu = [2\mu_1, 2\mu_2, \ldots, 2\mu_t]$ where $2\mu \vdash 2n$.

Observation: There exists a bijection between the set of all partitions of n and the set of even partitions of $2n$.

Note: We use exponential notation to be concise. This means that

$$2\mu = [4, 2, 2] = [4, 2^2].$$

Building our graphs

Definition

Let P and Q be two perfect matchings in $M(K_{2n})$ and $\mu = [\mu_1, \mu_2, \ldots, \mu_t]$ is a partition of n. We say that P and Q are μ-related if $P \cup Q = C_{2\mu_1} \cup C_{2\mu_2} \cup \cdots \cup C_{2\mu_t}$.

Example:

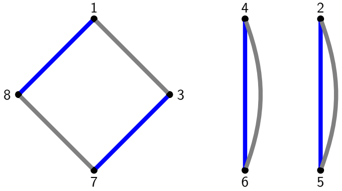

Figure: Our blue and grey perfect matching are $[2, 1^2]$-related.

Perfect matching association schemes

Definition

Let $\mu \vdash n$. The graph X_μ has vertex set $M(K_{2n})$. Two vertices of X_μ are adjacent if the union of the two corresponding matchings has cycle-type 2μ.

Definition

Let A_μ be the adjacency matrix of X_μ. The set

$$\mathcal{A}_{2n} = \{A_{[1^n]}, A_{[2,1^{n-2}]}, A_{[2,2,1^{n-4}]}, \ldots, A_{[n]}\}$$

is known as the perfect matching association scheme.

On the spectral gap

Problem

What is the second largest eigenvalue of each graph in the perfect matching association scheme?

Observation: The set $\mathcal{A}_{2n} = \{A_{[1^n]}, A_{[2,1^{n-2}]}, A_{[2,2,1^{n-4}]}, \ldots, A_{[n]}\}$ is a set of symmetric matrices that pairwise commute.

Fact: A set of symmetric matrices that pairwise commute have the same eigenspaces.

Perfect matching association scheme

• Let H_n be the subgroup of S_{2n} that is the stabilizer of a single perfect matching ($H_n = S_2 \wr S_n$):

$$|H_n| = 2^n \cdot (n!) \to [S_{2n} : H_n] = (2n-1)!!.$$

• We have a bijection between cosets of H_n and $M(K_{2n})$.
• S_{2n} acts on the set of cosets via right multiplication.
• The permutation matrices arising from this action is the induced representation $1\uparrow_{H_n}^{S_{2n}}$:

$$1\uparrow_{H_n}^{S_{2n}} = \oplus_{\lambda \vdash n} S^{2\lambda}.$$

Perfect matching association scheme

- The group S_{2n} acts on the set of pairs of cosets of $H_n = S_2 \wr S_n$.

- The orbits of this action are called orbitals.

- Two pairs of cosets are in the same orbit if the corresponding pairs of perfect matching have the same cycle structure μ.

- If each pair of cosets represent the edge of a graph, then the binary matrix arising from the orbital indexed by μ is A_μ.

Eigenspaces

• Matrices in $\mathcal{A}_{2n}$ commute with the permutation matrices in $1\uparrow_{H_n}^{S_{2n}}$.

• The eigenspaces of our matrices correspond to irreducible representations of the symmetric group S_{2n} that appear in the decomposition of $1\uparrow_{H_n}^{S_{2n}}$.

Each eigenspace is indexed by an even partition of $2n$.

Eigenspaces

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$					
$[2n-2,2]$					
$[2n-4,4]$					
$\vdots$					
$[2^n]$					

Eigenvalues

Question: Given a S_{2n}-module corresponding to λ, what is the eigenvalue of A_μ corresponding to this eigenspace?

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	?	?	?		?
$[2n-2,2]$	?	?	?		?
$[2n-4,4]$	?	?	?		?
$\vdots$	?	?	?		?
$[2^n]$	?	?	?		?

Notation: Let ϕ_μ^λ be the eigenvalue of the λ-eigenspace of A_μ.

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	?	?		?
$[2n-2,2]$	1	?	?		?
$[2n-4,4]$	1	?	?		?
$\vdots$	1	?	?		?
$[2^n]$	1	?	?		?

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	?	?		?
$[2n-4,4]$	1	?	?		?
$\vdots$	1	?	?		?
$[2^n]$	1	?	?		?

The eigenvalues of the $[2n]$-eigenspace corresponds to the degree of each graph (each graph is regular).

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	✓	✓	✓	✓
$[2n-4,4]$	1	?	?		?
$\vdots$	1	?	?		?
$[2^n]$	1	?	?		?

MacDonald (1979) gives formulas for the eigenvalues corresponding to the $[2n-2,2]$-eigenspace.

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	✓	✓	✓	✓
$[2n-4,4]$	1	✓	?		?
$\vdots$	1	✓	?		?
$[2^n]$	1	✓	?		?

Diaconis and Holmes (2002) determine all eigenvalues of $A_{[4,2,2,\ldots,2]}$.

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	✓	✓	✓	✓
$[2n-4,4]$	1	✓	?		✓
$\vdots$	1	✓	?		✓
$[2^n]$	1	✓	?		✓

MacDonald (1979) provides a formula for computing eigenvalues of $A_{[2n]}$.

Obvious approach

An obvious approach is to construct an eigenvector w from the λ-eigenspace and evaluate $A_\mu w$.

Lemma (Godsil and Meagher, 2015)

Let $H_n = S_2 \wr S_n$ and $x_\lambda \in S_{2n}$ such that $(H_n, x_\lambda H_n)$ is a pair of cosets in the λ-orbital of H_n. Then

$$\phi_\mu^\lambda = \frac{v_\mu}{2^n(n!)} \sum_{h \in H_n} \chi^\lambda(x_\mu h).$$

This approach involves evaluating a sum of irreducible characters in a coset of H_n.

Small cases

$[1^4]$	$[2,1^2]$	$[2^2]$	$[3,1]$	$[4]$	Rep.	Dim.
1	12	12	32	48	$[8]$	1
1	5	-2	4	-8	$[6,2]$	20
1	2	7	-8	-2	$[4^2]$	14
1	-1	-2	-2	4	$[4,2^2]$	56
1	-6	3	8	-6	$[2^4]$	14

Table: Eigenvalues of $\mathcal{A}_8$

By implementing Srinivasan's Maple code in Sage, we can obtain all eigenvalues of the perfect matching association scheme for $n \leqslant 15$.

Conjecture

Problem

On which eigenspace does the second largest eigenvalue occurs?

It is well-known that the largest eigenvalue occurs on the $[2n]$-eigenspace for each A_μ and that this eigenvalue is the degree of A_μ.

Conjecture

If μ contains at least two parts of length 1, then the second largest eigenvalue of A_μ occurs on the $[2n-2,2]$-eigenspace.

Conjecture

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,1^{n-3}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1				
$[2n-4,2,2]$	1	✓			✓
$\vdots$	1	✓			✓
$[2^n]$	1	✓			✓

Using a computer, we can verify this conjecture for $2n \leqslant 30$.

Conjecture

Conjecture

If μ contains at least two parts of length 1, then the second largest eigenvalue of X_μ occurs on the $[2n-2,2]$-eigenspace.

Why do we require that μ contains at least two parts of length 1?

If μ has no parts of size one (μ is a derangement), then $\phi_\mu^{[2n-2,2]}$ is negative. (MacDonald, 1979)

Relations with one part of size one

$[1^5]$	$[2,1^3]$	$[2^2,1]$	$[3,1^2]$	[3,2]	[4,1]	[5]	Space	Dim.
1	20	60	80	160	240	384	[10]	1
1	11	6	26	-20	24	-48	[8,2]	35
1	6	11	-4	20	-26	-8	[6,4]	90
1	3	-10	2	-4	-8	16	$[6,2^2]$	225
1	0	5	-10	-10	10	4	$[4^2,2]$	252
1	-4	-3	2	10	6	-12	$[4,2^3]$	300
1	-10	15	20	-20	-30	24	$[2^5]$	42

Table: Eigenvalues of $\mathcal{A}_{10}$

Results

Theorem (GHLMM (2025+))

Let $\mu = [n-k, \mu']$ with $\mu' \vdash k$. If n is sufficiently large relative to k, then $\phi_\mu^{[2n-2,2]}$ is the second largest eigenvalue of X_μ in absolute value.

Theorem (GHLMM (2025+))

If

$$\mu \in \{[2,1^{n-2}],[3,1^{n-3}],[2^2,1^{n-4}],[4,1^{n-4}],[3,2,1^{n-5}],[5,1^{n-5}]\}$$

then $\phi_\mu^{[2n-2,2]}$ is the second largest eigenvalue of X_μ in absolute value.

An inductive algorithm

- Srinivasan (2020) derived an inductive algorithm that allows us to obtain closed form formulas for the spectrum of X_μ based on content-evaluating symmetric functions.

An inductive algorithm

- Srinivasan (2020) derived an inductive algorithm that allows us to obtain closed form formulas for the spectrum of X_μ based on content-evaluating symmetric functions.

- Namely, Srinivasan shows that elements of the algebra of symmetric functions in $2n$ variables over $\mathbb{Q}[t]$ can be used to obtain closed-form formulae for the spectrum of X_μ.

Example

Example: Let $\phi^{\lambda}_{[3,1^{n-2}]}$ be the eigenvalue of $X_{[3,1^{n-2}]}$ occurring on the λ-eigenspace and let

$$p_1(x_1, x_2, \ldots x_{2n}) = \sum_{i=1}^{2n} x_i; \quad p_2(x_1, x_2, \ldots x_{2n}) = \sum_{i=1}^{2n} x_i^2.$$

Generating content

Let λ be an even partition of $2n$ that indexes an eigenspace. How do we generate the content of the Young tableau associated with λ, denoted $c(\lambda)$?

x_1	x_2	x_3	x_4	x_5	x_6
x_7	x_8	x_9	x_{10}		
x_{11}	x_{12}				

(a) Assignment of $2n$ variables to the boxes of a Young tableau for $n = 6$.

0	1	2	3	4	5
-1	0	1	2		
-2	-1				

(b) Content of Young tableau corresponding to the partition [6, 4, 2].

$$p_1(c(\lambda)) = \sum_{i=1}^{2n} x_i = 9; \quad p_2(c(\lambda)) = \sum_{i=1}^{2n} x_i^2 = 66.$$

Formula for spectrum

How do we piece together these symmetric functions to compute the spectrum of $X_{[3,1^{n-1}]}$?

$$\phi^{\lambda}_{[3,1^{n-2}]} = \frac{p_2(c(\lambda))}{2} - p_1(c(\lambda)) + \frac{3n - n^2}{4}$$

and thus

$$\phi^{[6,4,2]}_{[3,1^{n-2}]} = \frac{66}{2} - 9 + \frac{3(3) - (3)^2}{4} = 24.$$

Application to second largest eigenvalue

How can we use these formulae to show that $\phi^{[2n-2,2]}_{[3,1^{n-1}]}$ is the second largest eigenvalue?

- Every even partition of $(2n+2)$ can be obtained from an even partition of $2n$, λ, by adding two boxes to a row of the Young tableau.

Application to second largest eigenvalue

How can we use these formulae to show that $\phi_{\mu}^{[2n-2,2]}$ is the second largest eigenvalue?

- Every even partition of $(2n+2)$ can be obtained from an even partition of $2n$, λ, by adding two boxes to a row of the Young tableau.

0	1	2	3	4	5
−1	0	1	2		
−2	−1				
−3	−2				

(a) Young tableau for partition $2\lambda = 2[3,2,1^2]$ with its content.

0	1	2	3	4	5
−1	0	1	2		
−2	−1	0	1		
−3	−2				

(b) Young tableau for partition $2\lambda^{+} = 2[3,2^2,1]$ with its content.

Induction

Induction hypothesis: We assume that $\phi^{[2n-2,2]}_{[3,1^{n-1}]}$ is the second largest for $2n$.

0	1	2	3	4	5	6	7	8	9	10
−1	0									

(a) Young tableau for partition $2\lambda = 2[5,1]$ and $2n = 12$.

Induction

Induction step: We compute

$$\phi^{[2n,2]}_{[3,1^n]} - \phi^{[2n-2,2]}_{[3,1^n]} = 4n^2 - 12n + 6.$$

0	1	2	3	4	5	6	7	8	9	10
−1	0									

0	1	2	3	4	5	6	7	8	9	10	11	12
−1	0											

Figure: Illustrating change in content of Young tableaux.

Induction

Key step: Show that

$$\phi^{\lambda}_{[3,1^{n-1}]} - \phi^{\lambda^+}_{[3,1^n]} < 4n^2 - 12n + 6$$

when $\lambda \notin \{[2n], [2n-1]\}$.

Since the increase of each eigenvalue does not exceed the increase seen in $\phi^{[2n-2,2]}_{[3,1^{n-1}]}$, by the induction hypothesis, $\phi^{[2n,2]}_{[3,1^n]}$ must also be the second largest eigenvalue.

Other formulae

A_μ	E_μ
$A_{[2,1^{n-2}]}$	$\frac{p_1}{2} - \frac{t}{4}$
$A_{[3,1^{n-3}]}$,	$\frac{p_2}{2} - p_1 + \frac{3t-t^2}{4}$
$A_{[2,2,1^{n-4}]}$	$\frac{p_1^2}{8} - \frac{3p_2}{4} + \frac{(10-t)p_1}{8} + \frac{9t^2-24t}{32}$
$A_{[4,1^{n-4}]}$	$\frac{p_3}{2} - \frac{9p_2}{4} + \frac{(11-2t)p_1}{2} + \frac{8t^2-23t}{8}$
$A_{[3,2,1^{n-5}]}$	$-2p_3 + \frac{1}{4}p_1p_2 + (\frac{60-t}{8})p_2 - \frac{1}{2}p_1^2 + \frac{29t-120-t^2}{8}p_1 + \frac{116t-47t^2+t^3}{16}$
$A_{[5,1^{n-5}]}$	$\frac{p_4}{2} - 4p_3 + \frac{40-3t}{2}p_2 - p_1^2 + (7t-34)p_1 + \frac{217t-96t^2+5t^3}{12}$

Table: Formulae for the symmetric functions to compute eigenvalues of certain matrices in the perfect matching association scheme

Result

Theorem (GHLMM (2025+))

If

$$\mu \in \{[2, 1^{n-2}], [3, 1^{n-3}], [2^2, 1^{n-4}], [4, 1^{n-4}], [3, 2, 1^{n-5}], [5, 1^{n-5}]\}$$

then $\phi_\mu^{[2n-2,2]}$ *is the second largest eigenvalue of* X_μ.

Future work

- What are the diameters of the graphs in $\mathcal{A}(M_{2n})$?
- What is the chromatic number of graphs in $\mathcal{A}(M_{2n})$?
- Can our methods be further extended to confirm our conjecture on the second highest eigenvalue?

Thank you!

The 2026 Prairie Discrete Math Workshop:

- Set to take place on May 7th and 8th in Regina;
- Students and post-docs welcome! (Some travel funding may be available)